Int. J. Heat Mass Transfer. Vol. 6, pp. 999–1018. Pergamon Press 1963. Printed in Great Britain.

USE OF REFERENCE STATES IN PREDICTING TRANSPORT RATES IN HIGH-SPEED TURBULENT FLOWS WITH MASS TRANSFERS

ELDON L. KNUTH† and HARVEY DERSHIN‡

(*Received* 4 *January* 1963 *and in revised form* 3 *May* 1963)

Abstract—A method for predicting mass-, momentum-, and energy-transfer rates in turbulent flows is outlined involving (a) a semi-empirical correlation of friction coefficient for injection of air into a low-speed air stream (providing the "constant-property" curve forming the heart of any attempt to predict transfer rates using reference states), (b) modified Reynolds analogies relating mass- and energy-transfer coefficients with friction coefficient, and (c) expressions for reference temperature (or enthalpy) and reference composition. The blowing-rate parameter used in the friction-coefficient correlation differs from suggestions of earlier investigators using sublayer theories chiefly in that the geometric average of the friction factor with blowing and the friction factor without blowing is used. The modified Reynolds analogies differ from the analogies of Rubesin and Pappas chiefly in that the specific heat of the mixture is constant with value fixed by the reference state. Since, for flows without mass additions at the wall, the reference-temperature expression used for laminar flows correlates successfully the results for turbulent flows, attempts to correlate data for turbulent flows with mass additions using the reference-state expressions developed by Knuth for laminar flows with mass additions are recommended.

Since no data including dependable measurements of foreign-gas concentration at the surface are available, no verification of the modified Reynolds analogy relating mass and momentum transfer and of the use of the reference-concentration expression is possible at present. Limited available data indicate that (a) for Mach numbers up to 3, the reference-temperature expression developed for laminar flows with mass transfers appears to correlate satisfactorily the data for turbulent flows with mass transfers, and (b) use of the modified Reynolds analogy and the proposed blowing-rate parameter appears to correlate satisfactorily the skin-friction and heat-transfer data for the case in which the heat capacity of the coolant and main-stream gas are equal. The need for data which are more extensive and more accurate is emphasized.

NOMENCLATURE

- A, dimensionless laminar sublayer thickness [cf. equations (3–4)];
- B_f, blowing rate for momentum transfer (cf. Table 1);
- B_h, blowing rate for energy transfer (cf. Table 1);
- B_m, blowing rate for mass transfer (cf. Table 1);
- B_s, blowing rate based on sublayer thickness [cf. equation (10)];
- $\frac{C_f}{2}$, $\frac{1}{2}\times$ friction coefficient $\equiv \frac{\tau_w}{\rho_\infty u_\infty^2}$;
- $\frac{C_{fm}}{2}$, $\frac{1}{2}\times$ friction coefficient based on maximum shearing stress $\equiv \frac{\tau_m}{\rho_\infty u_\infty^2}$;
- C_h, Stanton number $\equiv \frac{k_w(\partial T/\partial y)_w}{(\rho u c_p)_\infty(T_r - T_w)}$;
- C_m, mass-transfer coefficient $\equiv \frac{(\rho v)_w(1 - c_w^c)}{(\rho u)_\infty(c_w^c - c_\infty^c)}$;§
- c^k, mass fraction of component k;
- c_p, specific heat at constant pressure;
- D, diffusion coefficient in Fick's Diffusion Law;
- h, specific enthalpy;
- k, thermal conductivity;

† Associate Professor of Engineering, Head of Molecular-Beam Laboratory, University of California, Los Angeles 24, California.

‡ Senior Research Engineer, General Dynamics Corporation, Pomona, California.

§ Definition of C_m given in [13] contains typographical error.

M, molecular weight;
Ma, Mach number;
Pr, Prandtl number $\equiv \frac{c_p \mu}{k}$;
r, temperature recovery factor $\equiv \frac{2c_{p\infty}(T_r - T_\infty)}{u_\infty^2}$;
R, effectiveness $\equiv (T_w - T_c)/(T_{ro} - T_c)$;
Re, Reynolds number $\equiv \rho u_\infty x/\mu$;
Sc, Schmidt number $\equiv \frac{\mu}{\rho D}$;
T, temperature;
T_c, temperature of coolant [cf. equation (26)];
T_r, temperature of wall for case in which temperature gradient vanishes at wall;
u, velocity parallel to wall;
v, velocity normal to wall;
x, co-ordinate parallel to wall;
y, co-ordinate normal to wall;
δ, thickness of boundary layer (subscript ∞) or sublayer (subscript s);
μ, dynamic viscosity coefficient;
ρ, density;
τ, shearing stress;
τ_m, maximum shearing stress.

Superscripts:
a, mixture component(s) (e.g., air) other than those components added at wall;
c, mixture component(s) (e.g., coolant) added at wall;
$*$, variable evaluated at reference state;
$+$, dimensionless variable.

Subscripts:
s, sublayer boundary;
t, turbulent portion of boundary layer;
w, wall;
o, limiting value as blowing rate approaches zero;
∞, outer edge of boundary layer.

INTRODUCTION

ENGINEERS have been attempting to predict transfer rates (either mass-, momentum-, or energy-) by inserting fluid properties evaluated at some "reference" state into equations established for flows of constant-property fluids ever since the first attempts to analyse and predict transfer rates in systems containing either temperature or composition gradients were made. These attempts to use reference states are motivated by the time and effort saved using this approach rather than computing exact solutions for the equations of motion with variable fluid properties. Contributions by von Kármán [1], Rubesin and Johnson [2], Tucker [3], Young and Janssen [4], Eckert [5], Monaghan [6], Sommer and Short [7], Rott [8] and Burggraf [9], for the case without composition gradients, are mentioned in the brief review of the effective-temperature hypothesis included in a recent report by Coles [10]; contributions by Scott [11] and Gross *et al.* [12], for the case with composition gradients, have been summarized and extended by Knuth [13]. In [13] the physical bases for the concepts of reference temperatures and reference compositions are examined; methods for calculating reference temperatures and reference compositions for the case of laminar boundary-layer flows with mass additions at the wall are developed (with the aid of analytical results for a Couette-flow model). The following principal conclusions are drawn:

1. The reference composition appropriate for use in relations developed for fluids with constant properties must be calculated, in general, using

$$c^{a*} = \frac{M^a}{M^a - M^c} \frac{\ln M_\infty/M_w}{\ln c^a_\infty M_\infty/c^a_w M_w}.$$

The linearized expression $c^{a*} = \frac{1}{2}(c^a_w + c^a_\infty)$ may be used only for low blowing rates or small molecular-weight differences.

2. The reference temperature appropriate for use in relations developed for fluids with constant properties may be calculated using either

$$T^* \approx 0{\cdot}5(T_w + T_\infty) + 0{\cdot}2r^*_o \frac{u_\infty^2}{2c_p^*} + 0{\cdot}1\left[B_h^* + (B_h^* + B_m^*)\frac{c_p^c - c_p^*}{c_p^*}\right](T_w - T_\infty)$$

or

$$h^* \approx 0{\cdot}5(h_w + h_\infty) + 0{\cdot}2r^*_o \frac{u_\infty^2}{2} + 0{\cdot}1\left[B_h^* + (B_h^* - 2B_m^*)\frac{c_p^c - c_p^*}{c_p^*}\right](h_w - h_\infty)$$

the latter equation to be used for cases in which variations of the heat capacities with temperature are significant. These analytical expressions are simple extensions of the empirical expression given by Eckert [5] for the reference temperature (or enthalpy) for the case of boundary-layer flows with no mass addition at the wall.

Since Eckert [5] found that, for flows without mass additions at the wall, the reference-temperature expression used for laminar flows correlated successfully the results for turbulent flows, one might be encouraged to attempt to correlate results for turbulent flows with mass additions using the reference-state expressions developed in [13] for laminar flows and modifying appropriately the blowing-rate parameters. The blowing-rate parameters used in the reference-state expressions for laminar Couette and boundary-layer flows are given in the second and third columns of Table 1. If sufficient information concerning turbulent flows were available, then one might develop empirically turbulent-flow analogs to these parameters.

Table 1. Comparison of blowing-rate parameters used in reference-state expressions

Blowing-rate parameter	Laminar Couette flow (analytical)	Laminar boundary layer (empirical)	Turbulent boundary layer (provisional)
B_f^*	$\frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_{fo}^*}$	$\frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_{fo}^*}$	$\frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_{fo}^*}$
B_m^*	$B_f^* \frac{C_{fo}^*}{2C_{mo}^*}$	$B_f^* \left(\frac{C_{fo}^*}{2C_{mo}^*}\right)^{(C_f^*/C_{fo}^*)^{1/6}}$	$B_f^* \frac{C_{fo}^*}{2C_{mo}^*}$
B_h^*	$B_f^* \frac{C_{fo}^*}{2C_{ho}^*}$	$B_f^* \left(\frac{C_{fo}^*}{2C_{ho}^*}\right)^{(C_f/C_{fo})^{1/6}}$	$B_f^* \frac{C_{fo}^*}{2C_{ho}^*}$

Unfortunately, sufficient information is not available. Hence, until the required information becomes available, one might use provisional blowing-rate parameters suggested by these results for laminar flows. Pertinent to the selection of such parameters is the observation that the difference in the forms given in the second and third columns appears to be related to the fact that the several transport rates for laminar Couette flow approach zero asymptotically as the blowing rate increases whereas the corresponding transport rates for laminar boundary-layer flow vanish simultaneously at a finite value of the blowing-rate parameter B_f^*. Available information indicates that, in this respect, turbulent boundary-layer flow (with its laminar sublayer flow) resembles laminar Couette flow more than laminar boundary-layer flow; the several transport rates approach zero asymptotically as the blowing rate increases. Note also that, since $C_{fo}^*/2C_{mo}^*$, $C_{fo}^*/2C_{ho}^*$ and $(C_f^*/C_{fo}^*)^{1/6}$ are frequently of the order of unity, the difference in the values of a given blowing-rate parameter computed using the forms given in the second and third columns of Table 1 is small in many applications. Hence, one might use provisionally the blowing-rate parameters given in the fourth column of Table 1. The present paper describes the results of an effort to correlate transfer rates, for turbulent flows with mass additions, using these parameters in the reference-state expressions developed in [13] for laminar flows.

In the case of laminar flows with mass additions [13] both the constant-property relations and the variable-property relations were supplied by results of exact analyses, the author believing that "a correlation of results of exact calculations would be just as meaningful as a correlation of experimental results". In the case of turbulent flows, the situation is quite different; the present state of knowledge of turbulent flows with mass transfer is such that one would question the value of any reference-state expressions obtained examining only analytical results. Hence, one is led to examine experimental results.

In order to establish whether or not the suggested reference-state expressions may be applied to turbulent flows with mass transfers, measured values of friction coefficients, mass-transfer coefficients, heat-transfer coefficients and recovery factors are required for wide ranges of Mach numbers, Reynolds numbers, and blowing rates; temperatures and concentrations must be known at the wall as well as at the outer edge of the boundary layer. Of those references which have come to the attention of the authors, only two [14] and [15] give results of measurements of concentrations. Skin friction was not measured in either case; suspicion is cast on the heat-transfer data of [14] by [16] and the scatter of the heat-transfer data of [15] discourages the authors from using them for the present purposes. Furthermore, Danberg [15] did not attempt to measure the concentration at the wall (and extrapolation of a concentration profile to obtain a wall concentration is difficult). Also, it is believed that the method used by Scott *et al.* [14], i.e. sucking a gas sample through a hole in the wall, measures a coolant concentration which is less than the coolant concentration at the wall. Hence, only the applicability of the reference-temperature expression to data obtained for cases in which coolant and mainstream gas have similar properties will be discussed in this paper.

The measurement of skin friction for the case of mass addition at the wall is difficult. Of the several instruments (drag balance, pitot tube, and hot-wire anemometer) which have been used, the authors believe that the hot-wire anemometer has given the most accurate results. Consequently, even though Goodwin [17] and Smith [18] have been the only investigators to use this instrument for cases of interest to this paper, only skin-friction data taken with a hot-wire anemometer will be examined here.

The measurement of heat-transfer for the case of mass addition is less difficult than is the measurement of skin friction. The main obstacle is usually the control or evaluation of all pertinent heat sources and sinks. Of those data which have come to the attention of the authors, it is believed that the data of Bartle and Leadon [19] are the most reliable. Their test chamber was designed to minimize effects of thermal radiations; influences of all other heat sources and sinks were evaluated. They adjusted the blowing rate so that a uniform plate temperature was realized.

The following plan of attack was used in the study described here. First, a semi-empirical relationship describing the dependence of skin friction on blowing rate was developed using the data of Goodwin [17] and Smith [18] for the case of flows of constant-property fluids (i.e. for the case of air injection into a low-Mach-number air stream). Then the reference-temperature expression of Knuth [13] and a modified Reynolds analogy were used in an effort to place the heat-transfer data of Bartle and Leadon [19] for free-stream Mach numbers of 2·0 and 3·2 on the aforementioned semi-empirical curve. If the low-speed skin-friction data and the high-speed heat-transfer data fall, to good approximation, on the same curve, then (especially since the high-speed data were obtained at two different Mach numbers) the probability that both the reference-temperature expression and the modified Reynolds analogy are appropriate is high. Whereas such an implicit verification of the reference-temperature expression and the modified Reynolds analogy is not, in principle, the ideal approach, it appears to be the required approach as a consequence of the limited amount of experimental data.

The "constant-property" curve may be difficult to establish experimentally. Its pursuance is motivated, however, by its technical applicability and the fact that it needs to be determined only once. If it is determined, then the heat-, mass- and momentum-transfer rates for turbulent flows with mass additions can be predicted for a wide range of free-stream and wall conditions using the "constant-property" curve, the reference-state expressions and the modified Reynolds analogies.

MOMENTUM TRANSFER (SKIN FRICTION)

As stated in the Introduction, the plan of attack is to attempt to establish a "constant-property" curve describing the dependence of skin friction on blowing rate for the case of constant-property fluids and then to attempt to develop a method for predicting all other transfer rates using this "constant-property" curve. In the present section of the paper, the results of an effort to establish such a "constant-property" curve are described.

In the case of the turbulent boundary layer without mass addition at the wall, engineers have found that momentum-transfer rates and (if the Prandtl number is near unity) energy-transfer rates can be predicted to an approximation sufficient for many applications by using a simple model in which the boundary layer is divided into a laminar sublayer and a turbulent outer layer. Consequently, an attempt is made here to extend this simple model to include the case of mass addition at the wall.

Consider a laminar sublayer (Fig. 1) characterized by the following features:

1. Gradients in the x-direction (parallel to the stationary wall) are negligible in comparison with gradients in the y-direction (normal to the stationary wall).
2. Fluid properties (i.e. viscosity and density) are constant. (In making predictions for flows with variable properties, one would evaluate properties at the appropriate reference state.)

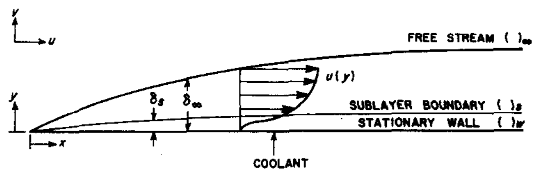

FIG. 1. Boundary-layer flow model for turbulent flow with mass addition at wall.

Then, within this laminar sublayer, conservation of momentum is described by the ordinary differential equation

$$\mu^* \frac{du}{dy} = \tau_w + (\rho v)_w u \qquad (1)$$

which states that the viscous force at an arbitrary plane equals the viscous force at the wall plus the force required to accelerate the mass added at the wall to velocity u. Separating variables, integrating from the stationary wall to the sub-layer boundary, and rearranging, one obtains

$$\frac{\tau_w \delta_s}{\mu^* u_s} = \frac{\dfrac{(\rho v)_w \delta_s}{\mu^*}}{\exp\left[\dfrac{(\rho v)_w \delta_s}{\mu^*}\right] - 1} \equiv \frac{B_s^*}{\exp(B_s^*) - 1}. \qquad (2)$$

The problem of predicting the viscous force at the wall for a given rate of mass addition at the wall and a given viscosity is reduced now to the problem of predicting values of the velocity u_s and distance δ_s corresponding to the sublayer boundary.

For the case with no mass addition at the wall, the dimensionless velocity

$$u_{so}^+ = \frac{u_s}{\sqrt{(\tau_{wo}/\rho^*)}} = A \text{ (a constant)} \qquad (3)$$

and the corresponding dimensionless distance

$$\delta_{so}^+ = \frac{\rho^* \sqrt{(\tau_{wo}/\rho^*)} \delta_s}{\mu^*} = A \qquad (4)$$

are found to describe the sublayer boundary to good approximation. For the case with mass addition at the wall, one would like to establish appropriate extensions of these dimensionless parameters. Reasonable requirements include:

1. The extensions must incorporate the fact that the viscous stress varies throughout the sublayer (monotonically from a minimum at the stationary wall to a maximum at the sublayer boundary).
2. The extensions, taken together, must satisfy (at least to a satisfactory approximation) equation (2).
3. In the absence of information favoring dissimilar modifications of u_{so}^+ and δ_{so}^+, it is rational to modify the two parameters similarly.
4. The extensions must reduce, for zero blowing rate, to the expressions quoted for the case with no mass addition at the wall, i.e. to equations (3) and (4).

Equation (2) may be rearranged, keeping in mind that

$$\frac{\tau_w}{\tau_s} = e^{-B_s^*}, \qquad (5)$$

to obtain

$$\frac{\tau_w \delta_s}{\mu^* u_s} = e^{-\frac{1}{2}B_s^*} \frac{B_s^*}{e^{\frac{1}{2}B_s^*} - e^{-\frac{1}{2}B_s^*}}$$

$$= \left(\frac{\tau_w}{\tau_s}\right)^{1/2} \frac{\frac{1}{2}B_s^*}{\sinh \frac{1}{2}B_s^*}$$

or

$$\frac{\sqrt{(\tau_w \tau_s)}\delta_s}{\mu^* u_s} = \frac{\frac{1}{2}B_s^*}{\sinh \frac{1}{2}B_s^*} \approx 1 - \frac{1}{24} B_s^{*2} + \dots .$$

If B_s^* is smaller than approximately 2 (small to modest blowing rates), then this expression may be put in the form

$$\frac{u_s}{\sqrt{[(\tau_s \tau_w)^{1/2}/\rho^*]}} \approx \frac{\rho^* \sqrt{[(\tau_s \tau_w)^{1/2}/\rho^*]}\delta_s}{\mu^*}.$$

Hence, requirement 1 is satisfied qualitatively, requirement 2 is satisfied up to and including first-order terms in blowing rates, and requirements 3 and 4 are satisfied exactly, by the extensions

$$u_s^+ = \frac{u_s}{\sqrt{[(\tau_w \tau_s)^{1/2}/\rho^*]}} = A \tag{6}$$

$$\delta_s^+ = \frac{\rho^* \sqrt{[(\tau_w \tau_s)^{1/2}/\rho^*]}\,\delta_s}{\mu^*} = A \tag{7}$$

i.e. by replacing the viscous stress at the wall for the case with no mass transfer by the geometric average of the viscous stress at the wall and the viscous stress at the sublayer boundary for the case with mass addition at the wall. The parameter B_s^* appearing in equation (5) is given now by

$$B_s^* \approx A \frac{(\rho v)_w}{\rho^* u_\infty} \left(\frac{\rho^* u_\infty^2}{\tau_w} \frac{\rho^* u_\infty^2}{\tau_s} \right)^{1/4}. \tag{8}$$

The exponential dependence shown in (5) has been derived previously; the form of the blowing-rate parameter is new.

Comparisons of (5) and (8) with data require information concerning the value of the viscous stress at the sublayer boundary and evaluation of the constant A. The most reliable measurements of variation of viscous stress within the boundary layer for the case with mass addition at the wall are perhaps the measurements of Smith [18]. From an examination of these data, the viscous stress at the sublayer boundary appears to be equal to or slightly less than the maximum shearing stress within the boundary layer; the value of the constant A appears to be about 11·5, the value proposed by von Kármán [20] for zero mass injection. Substituting the measured maximum stress τ_m for the viscous stress τ_s and setting $A = 11·5$, the correlation shown in Fig. 2 results. The agreement of the data with (5) and (8) is excellent.

In a typical application of (5) and (8) for the prediction of values of the viscous stress τ_w, measured values of the maximum stress τ_m are not available. Furthermore, analytical predictions are not available either. Fortunately, examinations of available data on distributions of viscous stresses within boundary layers reveal that the viscous stress at the sublayer boundary for the case with mass addition is of the order of magnitude of (but not necessarily equal to) the viscous stress at the wall for the case with no mass addition. Furthermore, one would expect that the ratio τ_s/τ_{wo} of these two viscous stresses would be a function of the

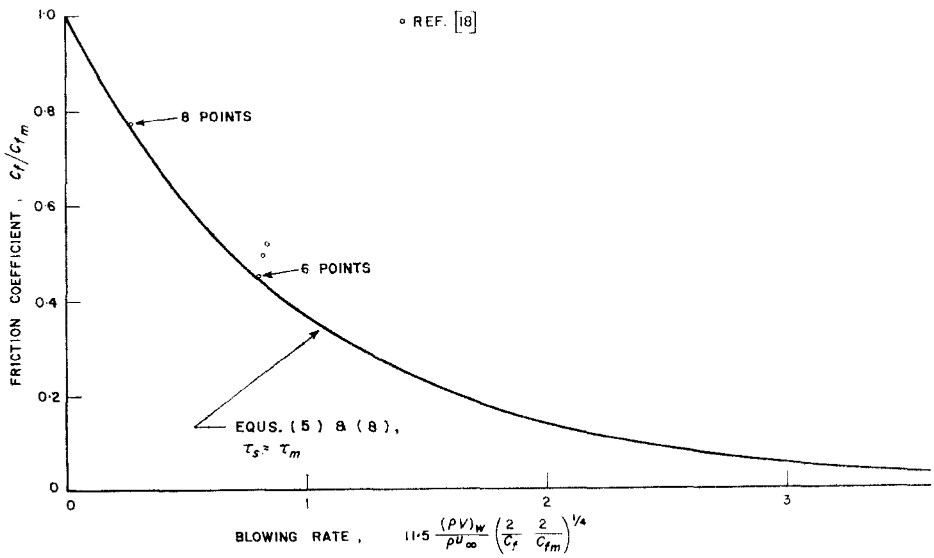

FIG. 2. Friction coefficient for injection of air into low-speed air stream as function of blowing rate (viscous stress τ_s replaced by maximum stress τ_m).

blowing rate B_s^*. Hence, one is led to an attempt to correlate the skin-friction data plotting

$$\frac{C_f}{C_{fo}} = f(B_s^*) \qquad (9)$$

with

$$B_s^* \approx 11{\cdot}5 \frac{(\rho v)_w}{\rho^* u_\infty}\left(\frac{2}{C_f^*}\,\frac{2}{C_{fo}^*}\right)^{1/4}. \qquad (10)$$

To the extent that the ratios τ_s/τ_{wo} and $(B_s^*/2)/(\sinh B_s^*/2)$ deviate from unity, one would expect that the functional dependence indicated in (9) would deviate from the exponential dependence indicated in (5). The blowing-rate parameter used here differs from suggestions of earlier investigators using sublayer theories chiefly in that the zero-blowing factor $(2/C_{fo})^{1/2}$ used, e.g. by Rannie [21] and by Knuth [22] and the arithmetic average

$$\frac{1}{2}\left[\left(\frac{2}{C_{fo}}\right)^{1/2} + \left(\frac{2}{C_f}\right)^{1/2}\right]$$

used by Turcotte [23] and Nash [24], is replaced here by the geometric average

$$\left[\left(\frac{2}{C_{fo}}\right)^{1/2}\left(\frac{2}{C_f}\right)^{1/2}\right]^{1/2}$$

motivated by the aforementioned requirement 3.

Using the parameters indicated in (9) and (10), the low-speed skin friction data of Goodwin [17] and Smith [18] are plotted in Fig. 3. (The scatter in Smith's data appearing here may be due in part to the fact that the friction coefficients C_f and C_{fo} were measured by different investigators, Smith and Goodwin, in different tests. The friction coefficients C_f and C_{fm} used in Fig. 2 were measured by one investigator in one test.) Differences between a curve faired through these points and the exponential curve $C_f/C_{fo} = e^{-B_s}$ may be due in part to deviations of the ratio τ_s/τ_{wo} from unity. Until more extensive skin-friction data for the case of constant-property fluids are available, Fig. 3 will be considered a provisional "constant-property" representation.

MASS AND ENERGY TRANSFERS (MODIFIED REYNOLDS ANALOGIES)

Recall that the purpose of the study described here is to attempt to predict transport rates using (a) a "constant-property" curve obtained from low-speed skin-friction measurements, (b) modified Reynolds analogies relating mass- and energy-transfer coefficients with skin-friction

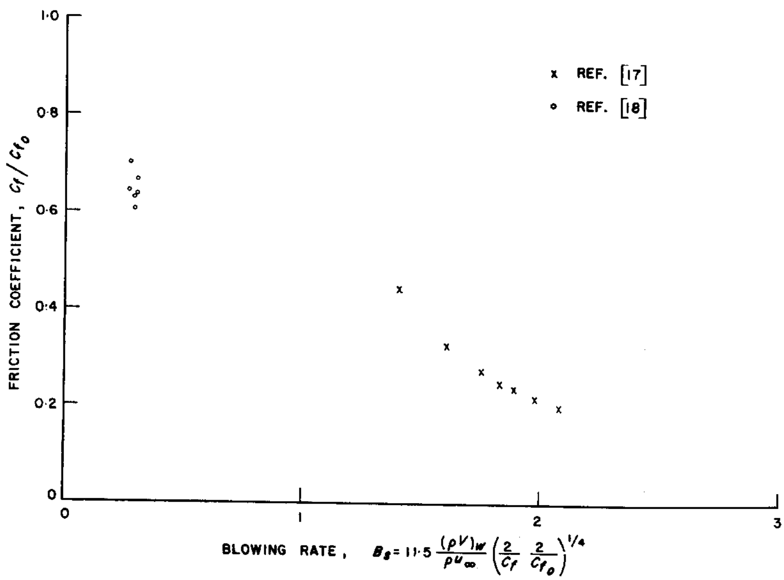

FIG. 3. Friction coefficient for injection of air into low-speed air stream as function of blowing rate (viscous stress τ_s replaced by wall stress τ_{wo} for case of no mass addition).

coefficients, and (*c*) reference-temperature and reference-composition expressions. In the Introduction, use of the reference-state expressions developed for laminar flows with mass transfers was recommended. In the section titled Momentum Transfer (Skin Friction), a provisional "constant-property" curve was presented. In the present section, the required modified Reynolds analogies are derived and compared with available data.

The following treatment of mass and energy transfer is similar to the treatment of Rubesin and Pappas [25] in that modified Reynolds analogies relating mass, momentum and energy transfers are derived neglecting gradients parallel to the stationary surface; it differs from the work of Rubesin and Pappas chiefly in that the specific heat of the mixture is constant with value fixed by the reference state. It is intended that the modified Reynolds analogies derived here be used in connection with the friction-coefficient expression developed in the preceding section in order to predict mass- and heat-transfer rates.

Consider the turbulent Couette-flow model (Fig. 4) characterized by the following features:

1. The velocity of the moving surface, as well as the temperature and concentrations at this surface, are uniform and steady, and are specified.
2. Heat and mass may pass readily through the moving surface; a steady force, required to maintain steady motion, acts on this surface in the direction of motion.
3. The momentum flux and the viscous stress in the direction normal to the two surfaces are much smaller than the pressure at some reference plane in the model.
4. The kinetic energy associated with the mass-weighted average velocity in the direction normal to the two surfaces is much smaller than the enthalpy of the fluid at some reference plane in the model.
5. Fick's Diffusion Law describes to good approximation the diffusion of the gas added at the wall relative to the rest of the mixture.
6. Body force, Dufour, and Soret effects are negligible.
7. The laminar Prandtl, Schmidt, and Lewis numbers, as well as the turbulent Prandtl, Schmidt, and Lewis numbers, are constants not equal to unity.
8. The mixture density, mixture heat capacity, coolant heat capacity, laminar transport coefficients, and turbulent transport coefficients may be treated as constants. (Treating the mixture heat capacity as a constant, with value fixed by the reference state, provides, in comparison with the laminar-flow analysis of [13], a simplification tending to compensate for complications introduced by considerations of turbulence.)

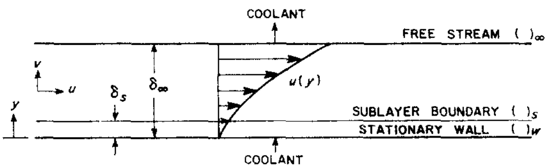

FIG. 4. Couette flow model for turbulent flow with mass addition at wall.

This simple model contains the most important physical features of a turbulent high-speed boundary layer involving mass, momentum and energy transfers with arbitrary laminar and turbulent Prandtl, Schmidt and Lewis numbers and with heat capacity of coolant differing from heat capacity of main-stream gas.

Integrating the appropriate ordinary differential equations describing conservations of mass, momentum and energy, one obtains, after considerable algebraic manipulation (cf. Appendix),

$$1 + \frac{(\rho v)_w}{\rho^* u_\infty} \frac{1}{C_m^*} = \left[1 + \frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_f^*} \frac{u_s}{u_\infty}\right]^{Sc^* - Sc_t} \times \left[1 + \frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_f^*}\right]^{Sc_t} \quad (18)$$

and

$$1 + \frac{(\rho v c_p^c)_w}{\rho^* u_\infty c_p^*} \frac{1}{C_h^*} = \left[1 + \frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_f^*} \frac{u_s}{u_\infty}\right]^{(c_p^c/c_p^*)(Pr^* - Pr_t)} \times \left[1 + \frac{(\rho v)_w}{\rho^* u_\infty} \frac{2}{C_f^*}\right]^{(c_p^c/c_p^*) Pr_t} \quad (20)$$

with

$$\begin{aligned} r^* \equiv 1 - 2 & \frac{\dfrac{c_p^*}{c_p^c}\left[\dfrac{C_f^*}{2}\dfrac{\rho^* u_\infty}{(\rho v)_w}\right]^2}{\left(2 - \dfrac{c_p^c}{c_p^*}Pr_t\right)\left(2 - \dfrac{c_p^c}{c_p^*}Pr^*\right)} \left\{\left(2 - \frac{c_p^c}{c_p^*}Pr_t\right)\left[1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\frac{u_s}{u_\infty}\right]^{(c_p^c/c_p^*)(Pr^* - Pr_t)}\right. \\ & \times \left(1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\right)^{(c_p^c/c_p^*)\,Pr_t} + \frac{c_p^c}{c_p^*}(Pr_t - Pr^*)\left[1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\frac{u_s}{u_\infty}\right]^{2 - (c_p^c/c_p^*)\,Pr_t} \\ & \times \left[1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\right]^{(c_p^c/c_p^*)\,Pr_t} - \left(2 - \frac{c_p^c}{c_p^*}Pr^*\right)\left[1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\right]^2 + \left(2 - \frac{c_p^c}{c_p^*}Pr_t\right) \\ & \left. \times \left(2 - \frac{c_p^c}{c_p^*}Pr^*\right)\left[\frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\right]\left[1 + \frac{c_p^c + c_p^*}{2c_p^*}\frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\right]\right\}. \end{aligned} \tag{21}$$

Equations (18) and (20) are modified Reynolds analogies relating, respectively, mass and momentum transfers and energy and momentum transfers in turbulent flows; (21) is the corresponding expression for the temperature recovery factor. Equation (18) is essentially the same as equation (38) of Rubesin and Pappas [25]. Although (20) and (21) are slight extensions of

In an effort to determine how well these modified Reynolds analogies describe experimental results, one might substitute

$$1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\frac{u_s}{u_\infty} = \exp\,[(\rho v)_w \delta_s/\mu^*] = (\exp B_s^*)$$

into (18) and (20) and rearrange to obtain

$$\frac{C_f^*}{C_{fo}^*} = \frac{\dfrac{(\rho v)_w}{\rho^* u_\infty}\dfrac{2}{C_{fo}^*}}{\left[1 + \dfrac{(\rho v)_w}{\rho^* u_\infty}\dfrac{1}{C_m^*}\right]^{1/Sc_t}\exp\,[B_s^*(1 - Sc^*/Sc_t)] - 1} \tag{22}$$

$$\frac{C_f^*}{C_{fo}^*} = \frac{\dfrac{(\rho v)_w}{\rho^* u_\infty}\dfrac{2}{C_{fo}^*}}{\left[1 + \dfrac{(\rho v c_p^c)_w}{\rho^* u_\infty c_p^*}\dfrac{1}{C_h^*}\right]^{(c_p^*/c_p^c)(1/Pr_t)}\exp\,[B_s^*(1 - Pr^*/Pr_t)] - 1}. \tag{23}$$

equations (53) and (55) of Rubesin and Pappas in that the turbulent Prandtl and Schmidt numbers are considered to be arbitrary constants not necessarily equal to each other or to unity, they differ chiefly from the corresponding equations of Rubesin and Pappas in that the specific heat of the mixture is constant with value fixed by the reference state. As a consequence of these differences, the similarity in (18) and (20) is greater than in (38) and (55) of [25]. It is believed that this greater similarity facilitates the correlation of mass, energy and momentum transfers using the concept of a reference state.

Values of friction coefficients might be computed substituting measured values of other parameters into these two equations. (Since the blowing parameter, B_s^*, is a function of the friction coefficient, C_f^*, the computation of a friction coefficient from either of these equations might involve iterative procedures.) One might compare then these computed friction coefficients with measured friction coefficients.

Such computations have been made using the heat-transfer data of Bartle and Leadon [19] for nitrogen injection into an air stream with free-stream Mach numbers of 2·0 and 3·2. Setting $Pr_t = 1$ and $c_p^c/c_p^* = 1$, (23) becomes

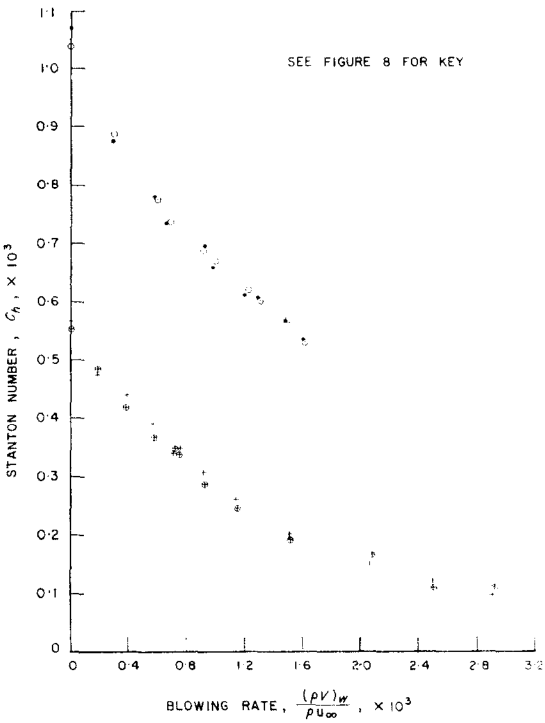

FIG. 5. Stanton number as function of blowing rate for two free-stream Mach numbers and four free-stream Reynolds numbers (fluid properties evaluated at free-stream temperature).

$$\frac{C_f^*}{C_{fo}^*} = \frac{\dfrac{(\rho v)_w}{\rho^* u_\infty} \dfrac{2}{C_{fo}^*}}{\left[1 + \dfrac{(\rho v)_w}{\rho^* u_\infty} \dfrac{1}{C_h^*}\right] \exp\left[B_s^*(1 - Pr^*)\right] - 1} \quad (24)$$

whereas, for $c_p^c = c_p^*$, the reference-temperature expression becomes

$$T^* = 0{\cdot}5(T_w + T_\infty) + 0{\cdot}2(T_{ro} - T_\infty) + 0{\cdot}1 \frac{(\rho v)_w}{\rho^* u_\infty} \frac{1}{C_{ho}^*} (T_w - T_\infty). \quad (25)$$

Figs. 5 and 6 illustrate the effect of evaluating fluid properties at the reference temperature; it is apparent that the large deviations from a common curve in Fig. 5 are due mostly to temperature effects. The deviations from a common curve remaining still in Fig. 6 are due mostly to Reynolds-number effects. Figs. 7 and 8 compare, as functions of the blowing rates B_f^* and B_s^*, the skin-friction coefficients computed from the heat-transfer data using (24) and evaluating fluid properties at reference temperatures given by (25), with the constant-property skin-friction coefficients presented previously in Fig. 3.

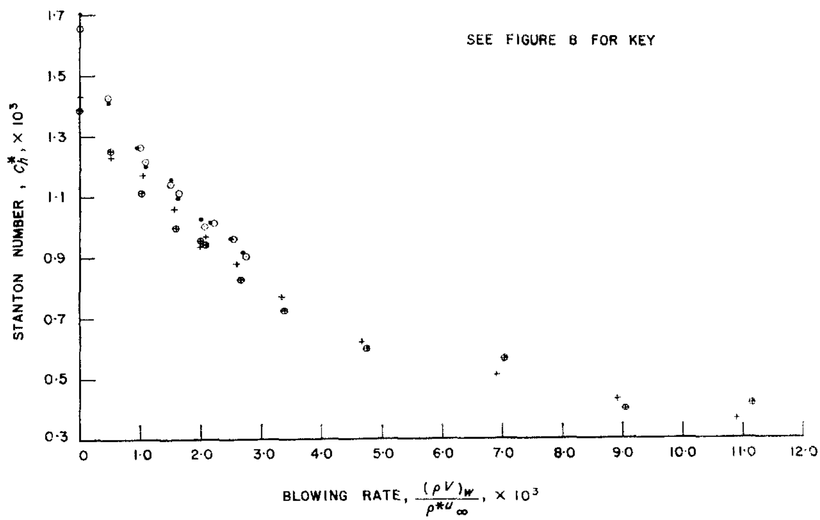

FIG. 6. Stanton number as function of blowing rate for two free-stream Mach numbers and four free-stream Reynolds numbers (fluid properties evaluated at reference temperature).

Fig. 7 is included since the abscissa B_f^* is essentially the same abscissa as used successfully in the correlation of skin-friction coefficients for laminar flows with mass transfers (cf. Table 1) and since several authors (e.g. [26]) have suggested the use of this parameter for turbulent flows with mass transfers. The distinct variations from a common curve found in Fig. 7 are similar to the variations from a common curve found in the plot of Stanton-number ratio C_h/C_{ho} vs. blowing rate B_h presented by Bartle and Leadon ([19], Fig. 6).

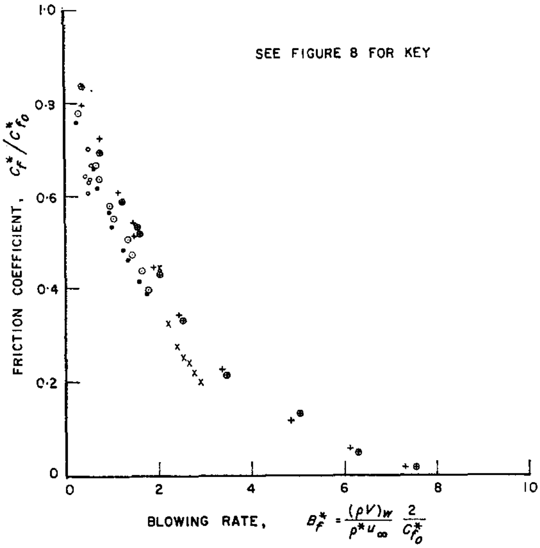

FIG. 7. Friction coefficient computed from high-speed heat-transfer data (using modified Reynolds analogy and reference temperature) compared with measured low-speed friction coefficient as function of blowing-rate parameter B_f^*.

The abscissa B_s^* used in Fig. 8 is motivated by the analysis of the present paper. Using this abscissa, both the measured low-speed skin-friction coefficients and the skin-friction coefficients computed from high-speed heat-transfer data using the modified Reynolds analogy and the suggested reference-temperature expression appear to be on a common curve within the limits of experimental error.

The empirical curve of Fig. 8 is compared in Fig. 9 with results of the theories of Rubesin [27] and Van Driest [28]. It is seen that, although the results of Rubesin's theory bracket the empirical curve, the theory predicts a significant dependence on Reynolds number which is not apparent in the available data. Results of Van Driest's theory, on the other hand, indicate a weaker dependence on Reynolds number (in better qualitative agreement with the available data)

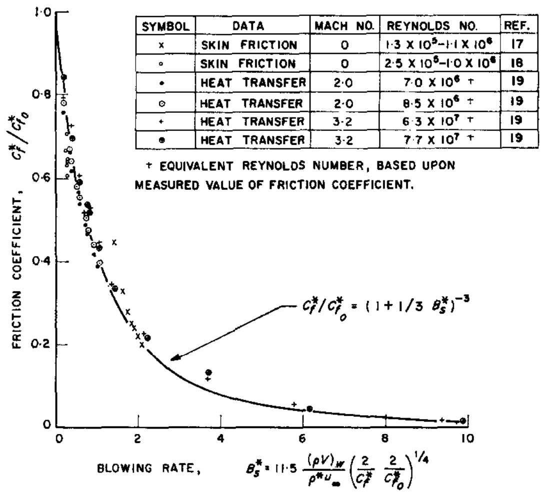

FIG. 8. Friction coefficient computed from high-speed heat-transfer data (using modified Reynolds analogy and reference temperature) compared with measured low-speed friction coefficient as function of blowing-rate parameter B_s^*.

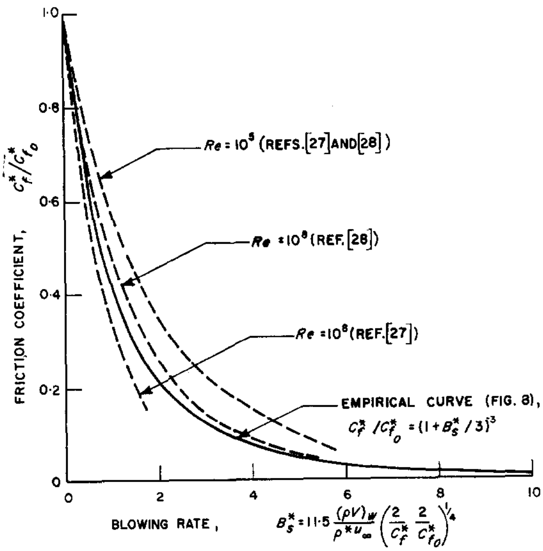

FIG. 9. Comparison of empirical friction-coefficient curve with results of theories for zero Mach number.

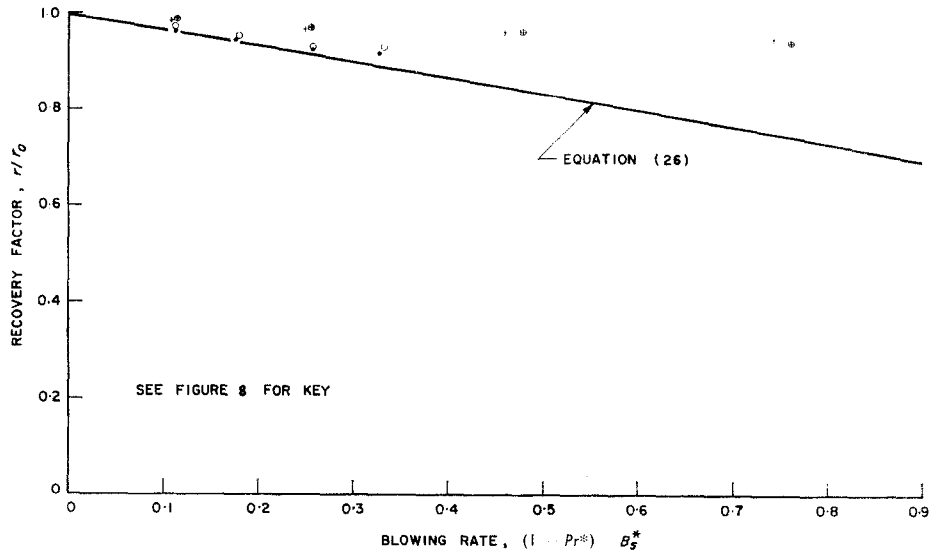

FIG. 10. Comparison of measured recovery factors with predictions of simplified linearized equation.

but do not predict skin-friction reductions as great as observed experimentally.

The attempt to correlate values of the recovery factor was less successful. The analytical result, (21), is cumbersome; experimental results are scarce. The current state of affairs is summarized in Fig. 10, where data obtained by Bartle and Leadon [19] for free-stream Mach numbers of 2·0 and 3·2 are compared with

$$\frac{r^*}{r_o^*} \approx 1 - \frac{1}{3}(1 - Pr^*)\,B_s^*. \tag{26}$$

[Equation (26) represents, to good approximation, the equation obtained by setting $Pr_t = 1$ and $c_p^c/c_p^* = 1$ in (21), expanding in series and retaining only first-order terms in blowing rates.] The cause for the failure of the data to form a single curve is not known.

If one wishes to use the effectiveness $R \equiv (T_w - T_c)/(T_{ro} - T_c)$ with temperature T_c defined by

$$k\left.\frac{\mathrm{d}T}{\mathrm{d}y}\right|_w \equiv (\rho v c_p^c)_w\,(T_w - T_c) \tag{27}$$

used successfully (for the range of test conditions investigated) by Bartle and Leadon [19], then, setting $Pr_t = 1$, (20) may be rearranged into

$$R = \frac{1}{1 + \left(\left\{\left[1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_f^*}\right]\left(\frac{\tau_s}{\tau_w}\right)^{Pr^* - 1}\right\}^{c_p^c/c_p^*} - 1\right)\frac{T_{ro} - T_w}{T_r - T_w}} \tag{28}$$

This equation is to be compared with the empirical equation

$$R = \frac{1}{\left[1 + \frac{1}{3}\frac{(\rho v c_p^c)_w}{(\rho u c_p)_\infty}\frac{1}{C_{ho}}\right]^3} \tag{29}$$

presented by Bartle and Leadon. In a comparison of these two equations, one might predict, using (28), values of the effectiveness R for the test conditions of [19] and compare these predicted values with measured values. Such predictions are most valuable to a designer if they can be made knowing only the test conditions (including blowing rate) and zero-blowing values of transport parameters (including friction coefficient and recovery factor).

Hence, values of C_f^*/C_{fo}^*, τ_w/τ_s, and r^*/r_o^*, required in evaluating (28), are to be predicted also. Although Fig. 8 is considered to be the best available correlation of friction coefficient with blowing rates, this figure cannot be used in the present calculation; most of the data used in preparing Fig. 8 are the same data with which results of the present calculation are to be compared. Therefore, although the procedure has been shown already to be inaccurate at high blowing rates (cf. Figs. 2 and 3), the ratios C_f^*/C_{fo}^* and τ_w/τ_s were approximated setting

$$\frac{\tau_w}{\tau_s} \approx \frac{C_f^*}{C_{fo}^*} \approx \exp(-B_s^*).$$

In the absence of established empirical relations, (26) was used in the required prediction of r^*/r_o^* so that

$$\frac{T_{ro} - T_w}{T_r - T_w} = \frac{1}{1 - \frac{1}{3}(1 - Pr^*)\frac{T_{ro} - T_\infty}{T_{ro} - T_w} B_s^*}. \quad (30)$$

Finally substituting into (28) and setting $c_p^c/c_p^* = 1$, the following expression, relating effectiveness with test conditions and zero-blowing values of transport parameters, was written:

$$R = \frac{1}{1 + \left\{\left[1 + \frac{(\rho v)_w}{\rho^* u_\infty}\frac{2}{C_{fo}^*}\exp(B_s^*)\right]\exp[(Pr^* - 1)B_s^*] - 1\right\}\left[1 - \frac{1}{3}(1 - Pr^*)\frac{T_{ro} - T_\infty}{T_{ro} - T_w}B_s^*\right]^{-1}} \quad (28a)$$

Values of the blowing rate B_s^* and the friction coefficient C_f^* were computed then solving simultaneously $(C_f^*/C_{fo}^*) = \exp(-B_s^*)$ and

$$B_s^* = 11 \cdot 5 \frac{(\rho v)_w}{\rho^* u_\infty}\left(\frac{2}{C_{fo}^*}\frac{2}{C_f^*}\right)^{1/4}$$

using values of $(\rho v)_w/(\rho u)_\infty$ and C_{fo} measured by Bartle and Leadon and values of T^* computed using (25). Substituting these values of B_s^* and measured values of the remaining parameters (i.e. test conditions) into (28a) and evaluating all fluid properties at the reference temperature T^*, the predicted values of effectiveness R plotted in Fig. 11 were computed. Deviations, at high blowing rates, of predicted values from the empirical (solid) curve are due mainly to the inaccuracy of

$$\frac{\tau_w}{\tau_s} \approx \frac{C_f^*}{C_{fo}^*} \approx \exp(-B_s^*).$$

The accuracy of future predictions of effectiveness R would be improved by replacing the exponential dependence of C_f^*/C_{fo}^* on B_s^* used in (28a) by the functional dependence indicated in Fig. 8.

These results suggest, for the test conditions of [19], an approximate equivalence of (28), derived analytically here, and (29), derived empirically by Bartle and Leadon. An examination of (28a) and (30) indicates, however, that the following three limitations might apply:

1. As emphasized by Tewfik [29], for T_c near T_{ro}, the value of R depends strongly upon the value of $(T_{ro} - T_w)/(T_r - T_w)$. Equation (30) indicates that a sufficient condition for weak dependence of R on these temperatures is given by

$$\left|\frac{1}{3}(1 - Pr^*)\frac{T_{ro} - T_\infty}{T_{ro} - T_w}B_s^*\right| \ll 1.$$

2. The apparent Mach-number independence of the blowing parameter

$$\frac{(\rho v)_w}{(\rho u)_\infty}\frac{1}{C_{ho}}$$

observed by Bartle and Leadon can be explained now if one notes that

$$\frac{(\rho v)_w}{(\rho u)_\infty} = \frac{(\rho v)_w}{\rho^* u_\infty}\frac{T_\infty}{T^*}$$

and that, to the extent that variations of heat capacities and Prandtl numbers with temperature may be neglected,

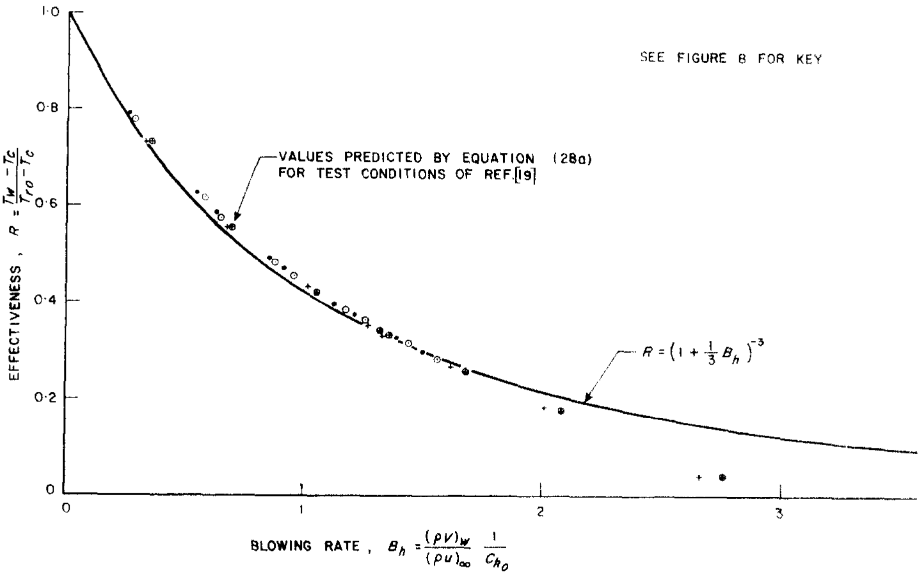

FIG. 11. Values of effectiveness R as function of blowing-rate parameter B_h as predicted by modified Reynolds analogy for test conditions of [19].

$$C_{ho} = C_{ho}(T_o^*) \frac{T_\infty}{T_o^*}$$

$$= C_{ho}(T^*) \frac{C_{ho}(T_o^*)}{C_{ho}(T^*)} \frac{T_\infty}{T_o^*}$$

$$= C_{ho}(T^*) \left[\frac{Re\,(T^*)}{Re\,(T_o^*)}\right]^{1/5} \frac{T_\infty}{T_o^*}$$

$$= C_{ho}(T^*) \left(\frac{T_o^*}{T^*}\right)^{0\cdot3} \frac{T_\infty}{T_o^*}$$

so that

$$\frac{(\rho v)_w}{(\rho u)_\infty} \frac{1}{C_{ho}} = \frac{(\rho v)_w}{\rho^* u_\infty} \frac{1}{C_{ho}^*} \left(\frac{T_o^*}{T^*}\right)^{0\cdot7}.$$

But, from (25),

$$\frac{T^*}{T_o^*} = 1 + \frac{(T_w - T_\infty)}{5(T_w + T_\infty) + 2(T_{ro} - T_\infty)} \frac{(\rho v)_w}{\rho^* u_\infty} \frac{1}{C_{ho}^*}.$$

For the test conditions of [19], this ratio is relatively insensitive to the value of the Mach number and it makes little difference whether one uses $(\rho v)_w/(\rho u)_\infty C_{ho}$ or $(\rho v)_w/\rho^* u_\infty C_{ho}^*$ in the correlations. In general, if $T^*/T_o^* \approx 1$, then $(\rho v)_w/(\rho u)_\infty C_{ho}$ may be used whereas, if T^*/T_o^* differs appreciably from unity, then fluid properties must be evaluated at the reference temperature T^*.

3. The apparent Reynolds-number independence of the correlation presented by Bartle and Leadon can be explained also if one notes that

$$\frac{C_{fo}}{2} = \frac{C_{fo}^*}{2} \left(\frac{T_o^*}{T^*}\right)^{0\cdot7} \frac{T^*}{T_\infty}.$$

For the test conditions of [19], C_{fo}^* varied only about 20 per cent. In general, and as indicated by Bartle and Leadon in Fig. 10 of [30], if C_{fo}^* varies significantly in a series of tests, then this variation will have to be considered in a correlation of the heat-transfer data.†

Additional experiments are required to establish

† The authors would like to acknowledge that computations suggested by E. R. Bartle led to the aforementioned explanation of the apparent Reynolds-number independence of the correlation presented by Bartle and Leadon.

in greater detail the conditions under which the concept of effectiveness is useful.

The expression for reference temperatures used here, i.e. (25), was motivated by the expression for reference temperature obtained by Knuth [13] for laminar flows with mass transfer and by the observation of Eckert [5] that the same reference-temperature expression correlates both laminar and turbulent flows for the case of no mass transfer. In an alternative motivation, one might extend the observations by Rott [8] and Burggraf [9] that, for turbulent flows without mass transfer, the reference temperature and the temperature at the sublayer boundary are equal to good approximation. Beginning with (15), an expression for the temperature T_s at the sublayer boundary has been developed retaining terms up to and including linear terms in blowing rates. The difference between the temperature T_s predicted by this expression and the temperature T^* predicted by (25) is so small that one is not able to decide in favor of one or the other of these equations on the basis of data available at present [31].

CONCLUSIONS

From examinations of existing data and analyses for turbulent boundary-layer flows with mass additions at the wall, the following conclusions have been drawn:

1. The present state of knowledge of turbulent flows with mass transfers is such that one would question the value of any reference-state expressions obtained examining only analytical results.
2. Available data are inadequate to ascertain whether the reference-composition expression developed for laminar flows with mass transfers is or is not applicable to turbulent flows with mass transfers.
3. Available data are inadequate to ascertain validities of the suggested (*a*) modified Reynolds analogies relating mass-transfer rate with momentum-transfer rate, (*b*) modified Reynolds analogy relating heat-transfer rate with momentum-transfer rate for the case in which the heat capacity of the coolant differs from the heat capacity of the main-stream gas, and (*c*) dependence of recovery factor on fluid properties, blowing rate, and viscous stress.
4. For the Mach-number range from 0 to 3, the reference-temperature expression developed for laminar flows with mass transfers appears to be adequate for correlating available data for turbulent flows with mass transfers.
5. Use of the modified Reynolds analogy and the proposed blowing-rate parameter appears to correlate satisfactorily the skin-friction and heat-transfer data for the case in which the properties of the coolant and main-stream gas are equal.
6. Additional turbulent skin-friction data are required for the case in which properties of the coolant and the main-stream gas are equal and temperature gradients are negligible. These data are necessary to establish more firmly the "constant-property" curve which forms the heart of any attempt to predict transfer rates using reference states.
7. Reliable and complete data for skin-friction are required for the case of injection of a foreign gas into a high-speed gas stream. Wide ranges of Mach numbers, Reynolds numbers and blowing rates are desirable; temperatures and concentrations must be known at the wall as well as at the outer edge of the boundary layer. These data would be useful in establishing whether or not the reference-state expressions developed for laminar flows with mass transfers are or are not applicable to turbulent flows with mass transfers over wide ranges of flow conditions.†
8. Reliable and complete data for heat transfer are required for the case of injection of a foreign gas. Since these data would be used primarily to ascertain validities of the suggested (*a*) modified Reynolds analogy relating heat-transfer rates with momentum-transfer rate for the case in which the heat capacity of the coolant differs from the heat capacity of the main-stream gas and (*b*) dependence of recovery factor on fluid properties, blowing rate, and viscous stress,

† A program to obtain such data is being undertaken presently by H. Dershin at General Dynamics, Pomona.

it follows that wide ranges of fluid properties, Reynolds number, and blowing rate are more important than is a wide range of Mach number. Temperatures and concentrations must be known at the wall as well as at the outer edge of the boundary layer.

9. Reliable and complete data for mass transfer are required for a wide range of fluid properties, blowing rate, and Reynolds number; temperatures and concentrations must be known at the wall as well as at the outer edge of the boundary layer. Although the required measurements would be included already in either of the experimental programs mentioned in Conclusions 7 and 8, one should not rule out the possibility of making these measurements independently of skin-friction or heat-transfer measurements.

Finally, the authors would appreciate the calling of their attentions to any data unknown to them at the present time and falling into one of the categories mentioned in Conclusions 6–9.

REFERENCES

1. T. VON KÁRMÁN, The problem of resistance in compressible fluids. Proceedings of the Fifth Volta Congress, Rome, 222–276 (1935).
2. H. W. RUBESIN and H. A. JOHNSON, A critical review of skin-friction and heat-transfer solutions of the laminar boundary layer of a flat plate. *Trans. A.S.M.E.* **71**, 383–388 (1949).
3. M. TUCKER, Approximate calculation of turbulent boundary-layer development in compressible flow. *NACA* Technical Note 2337, April (1951).
4. G. B. W. YOUNG and E. JANSSEN, The compressible boundary-layer, *J. Aero. Sci.* **19**, 229–236 (1952).
5. E. R. G. ECKERT, Engineering relations for friction and heat transfer to surfaces in high velocity flow. *J. Aero. Sci.* **22**, 585–587 (1955).
6. R. J. MONAGHAN, On the behaviour of boundary layers at supersonic speeds. Proceedings of the Fifth International Aeronautical Conference, Los Angeles, 277–315 (1955).
7. S. C. SOMMER and B. J. SHORT, Free-flight measurements of skin friction of turbulent boundary-layers with high rates of heat transfer at high supersonic speeds, *J. Aero. Sci.* **23**, 536–542 (1956).
8. N. ROTT, On the transfer of heat and mass in the turbulent boundary layer. Ramo-Wooldridge Corporation Report No. GM-TR-211, July (1957).
9. O. R. BURGGRAF, The compressibility transformation and the turbulent-boundary-layer equations, *J. Aero. Sci.* **29**, 434–439, April (1962).
10. D. E. COLES, The turbulent boundary layer in a compressible fluid. RAND Corporation Report No. R-403-PR, September (1962).
11. C. J. SCOTT, The application of constant property solutions to mass transfer cooling calculations. University of Minnesota, Engineering Memorandum No. 76, December (1958).
12. J. F. GROSS, J. P. HARTNETT, D. J. MASSON and CARL GAZLEY, JR., A review of binary laminar boundary-layer characteristics, *Int. J. Heat Mass Transfer*, **3**, 198–221 (1961).
13. E. L. KNUTH, Use of reference states and constant-property solutions in predicting mass-, momentum-, and energy-transfer rates in high-speed laminar flows, *Int. J. Heat Mass Transfer*, **6**, 1–22 (1963).
14. C. J. SCOTT, G. E. ANDERSON and D. R. ELGIN, Laminar, transitional and turbulent mass transfer cooling experiments at Mach numbers from 3 to 5. University of Minnesota, Institute of Technology Research Report No. 162 (AFOSR TN 59-1305, AD 231 974), August (1959).
15. J. E. DANBERG, Characteristics of the turbulent boundary layer with heat and mass transfer at Mach number 6·7. Proceedings of the Fifth United States Navy Symposium on Aeroballistics at the U.S. Naval Ordnance Laboratory, White Oak, Maryland, 16, 17, 18 October (1961).
16. E. R. BARTLE and B. M. LEADON, Experimental evaluation of heat transfer with transpiration cooling in a turbulent boundary layer at $M = 3{\cdot}2$, *J. Aero. Space Sci.* **27**, 78–80 (1960).
17. B. M. GOODWIN, The transpired turbulent boundary layer with zero pressure gradient. MIT Doctor of Science Thesis, May (1961).
18. K. A. SMITH, The transpired turbulent boundary layer. MIT Doctor of Science Thesis, May (1962).
19. E. R. BARTLE and B. M. LEADON, The compressible turbulent boundary layer on a flat plate with transpiration cooling I. Measurements of heat transfer and boundary layer profiles. Convair Scientific Research Laboratory Research Report No. 11, May (1961).
20. T. VON KÁRMÁN, Turbulence and skin friction. *J. Aero. Sci.* **1**, 1–20 (1934).
21. W. D. RANNIE, A simplified theory of porous wall cooling. Jet Propulsion Laboratory Progress Report No. 4-50, November (1947).
22. E. L. KNUTH, The mechanics of film cooling—Part 2. *Jet Propulsion* **25**, 16–25 (1955).
23. D. L. TURCOTTE, A sublayer theory for fluid injection into the incompressible turbulent boundary layer, *J. Aero. Space Sci.* **27**, 675–678, 711 (1960).
24. J. F. NASH, A correlation of skin-friction measurements in compressible turbulent boundary layers with injection. Aeronautical Research Council Report 22, 386, November (1960).
25. M. W. RUBESIN and C. C. PAPPAS, An analysis of the turbulent boundary-layer characteristics on a flat plate with distributed light-gas injection. *NACA* Technical Note 4149, February (1958).
26. B. M. LEADON, Comments on "A Sublayer Theory

for Fluid Injection", *J. Aero. Space Sci.* **28**, 826–827 (1961).
27. M. W. Rubesin, An analytical estimation of the effect of transpiration cooling on the heat-transfer and skin-friction characteristics of a compressible, turbulent boundary layer. *NACA* Technical Note 3341, December (1954).
28. E. R. Van Driest, On mass transfer near the stagnation point. North American Aviation Inc. Report No. AL-2553, June (1957).
29. O. E. Tewfik, On the effectiveness concept in mass-transfer cooling, *J. Aero. Space Sci.* **29**, 1382–1383 (1962).
30. E. R. Bartle and B. M. Leadon, The effectiveness as a universal measure of mass transfer cooling for a turbulent boundary layer. Proceedings of the 1962 Heat Transfer and Fluid Mechanics Institute, Seattle, 27–41.
31. E. L. Knuth, On two alternative motivations of reference-state expressions for turbulent flows with mass transfers, *AIAA J.* **1**, 1206–1208 (1963).

APPENDIX

For the turbulent Couette-flow model described in the section titled Mass and Energy Transfers (Modified Reynolds Analogies), conservations of mass, momentum and energy in the laminar sublayer are described by the ordinary differential equations

$$(\rho v)_w = -\rho^* D^* \frac{dc^c}{dy} + (\rho v)_w c^c \quad (11)$$

$$\tau_w = \mu^* \frac{du}{dy} - (\rho v)_w u \quad (12)$$

$$k^* \frac{dT}{dy}\bigg|_w = k^* \frac{dT}{dy} + \mu^* \frac{d(u^2/2)}{dy} - (\rho v)_w \left(h^c + \frac{u^2}{2} - h^c_w\right). \quad (13)$$

Eliminate the differential dy from (11) and (12) to obtain

$$\frac{d \ln (1 - c^c)}{d \ln [\tau_w + (\rho v)_w u]} = Sc^*$$

which may be integrated from the stationary wall to the sublayer boundary with the result

$$\frac{1 - c^c_s}{1 - c^c_w} = \left(\frac{\tau_s}{\tau_w}\right)^{Sc^*}. \quad (14)$$

This relation is to be combined later with a similar expression for the turbulent portion of the model in order to obtain the desired modified Reynolds analogy relating mass and momentum transfer.

Using the integrating factor $\exp[-(\rho v)_w c^c_p y/k^*]$. Equation (13) may be written

$$k^* \frac{dT}{dy}\bigg|_w = \mu^* \frac{d(u^2/2)}{dy} - (\rho v)_w \frac{u^2}{2} + k^* \exp[(\rho v)_w c^c_p y/k^*] \frac{d}{dy} (T - T_w) \exp[-(\rho v)_w c^c_p y/k^*].$$

Since

$$\mu^* \frac{d(u^2/2)}{dy} - (\rho v)_w \frac{u^2}{2} = \frac{\tau^2_w}{2(\rho v)_w} \times \{\exp[2(\rho v)_w y/\mu^*] - 1\}$$

the energy equation may be written

$$k^* \frac{dT}{dy}\bigg|_w = \frac{\tau^2}{2(\rho v)_w} \{\exp[2(\rho v)_w y/\mu^*] - 1\} + k^* \exp[(\rho v)_w c^c_p y/k^*] \frac{d}{dy} (T - T_w) \exp[-(\rho v)_w c^c_p y/k^*]$$

Multiplying by $\exp[-(\rho v)_w c^c_p y/k^*]$ and integrating from the stationary wall to the sublayer boundary

$$k^* (T_s - T_w) \exp[-(\rho v)_w c^c_p \delta_s/k^*)] = -k^* \frac{dT}{dy}\bigg|_w \frac{k^*}{(\rho v)_w c^c_p} \{\exp[-(\rho v)_w c^c_p \delta_s/k^*] - 1\}$$
$$- \frac{\tau^2_w}{2(\rho v)_w} \left[\frac{\exp\{[2(\rho v)_w \delta_s/\mu^*] - [(\rho v)_w c^c_p \delta_s/k^*]\} - 1}{\frac{2(\rho v)_w}{\mu^*} - \frac{(\rho v)_w c^c_p}{k^*}} + \frac{\exp[-(\rho v)_w c^c_p \delta_s/k^*] - 1}{\frac{(\rho v)_w c^c_p}{k^*}}\right]$$

which may be rearranged into

$$\frac{k^* \left.\dfrac{dT}{dy}\right|_w + (\rho v)_w c_p^c \left(T_s + r_s^* \dfrac{u_s^2}{2c_p^*} - T_w\right)}{k^* \left.\dfrac{dT}{dy}\right|_w} = \exp\{(c_p^c/c_p^*)\, Pr^* \,[(\rho v)_w \delta_s/\mu^*]\}$$

with

$$r_s^* \equiv \frac{c_p^*}{c_p^c}\left[\frac{\tau_w}{(\rho v)_w u_s}\right]^2 \frac{(c_p^c/c_p^*)\, Pr^* \left(\exp\{2\,[(\rho v)_w \delta_s/\mu^*]\} - 1\right) - 2\left(\exp\{(c_p^c/c_p^*)\, Pr^*\,[(\rho v)_w \delta_s/\mu^*]\} - 1\right)}{2 - (c_p^c/c_p^*)\, Pr^*}.$$

Combining with the similar expression for momentum transfer, i.e. with $(\tau_s/\tau_w) = \exp[(\rho v)_w \delta_s/\mu^*]$ one obtains

$$\frac{k^* \left.\dfrac{dT}{dy}\right|_w + (\rho v)_w c_p^c \left(T_s + r_s^* \dfrac{u_s^2}{2c_p^*} - T_w\right)}{k^* \left.\dfrac{dT}{dy}\right|_w} = \left(\frac{\tau_s}{\tau_w}\right)^{(c_p^c/c_p^*)\, Pr^*}. \tag{15}$$

This relation is to be combined later with a similar expression for the turbulent portion of the model in order to obtain the desired modified Reynolds analogy relating energy and momentum transfer.

Using turbulent Schmidt and Prandtl numbers in place of laminar Schmidt and Prandtl numbers, one obtains for the turbulent portion of the model the analogous relations

$$\frac{1 - c_\infty^c}{1 - c_s^c} = \left(\frac{\tau_\infty}{\tau_s}\right)^{Sc_t} \tag{16}$$

$$\frac{k^* \left.\dfrac{dT}{dy}\right|_s + (\rho v)_w c_p^c \left(T_\infty + r_\infty^* \dfrac{u_\infty^2}{2c_p^*} - T_s\right)}{k^* \left.\dfrac{dT}{dy}\right|_s} = \left(\frac{\tau_\infty}{\tau_s}\right)^{(c_p^c/c_p^*)\, Pr_t}. \tag{17}$$

Equations (16) and (17) are to be combined with (14) and (15) in order to obtain the desired modified Reynolds analogies.

Eliminating the coolant concentration c_s^c at the sublayer boundary from (14) and (16) one realizes for mass and momentum transfer

$$1 + \frac{c_w^c - c_\infty^c}{1 - c_w^c} = \left[1 + \frac{(\rho v)_w u_s}{\tau_w}\right]^{Sc^* - Sc_t} \left[1 + \frac{(\rho v)_w u_\infty}{\tau_w}\right]^{Sc_t}$$

or, using familiar dimensionless parameters, equation (18) of the text. In order to place the relations for energy and momentum transfer in similar form, both the temperature T_s and the temperature gradient $(dT/dy)|_s$ must be eliminated from (15) and (17). The third equation required to implement these eliminations is obtained by evaluating (13) at the sublayer boundary to obtain

$$k^* \left.\frac{dT}{dy}\right|_w = k^* \left.\frac{dT}{dy}\right|_s + \frac{\tau_w^2}{2(\rho v)_w} \times \left(\exp\{2\,[(\rho v)_w \delta_s/\mu^*]\} - 1\right) - (\rho v)_w c_p^c (T_s - T_w). \tag{19}$$

Eliminating now T_s and $(dT/dy)|_s$ from (15), (17), and (19) one realizes for energy and momentum transfer

$$1 + \frac{(\rho v)_w c_p^c \left(T_\infty + r^* \dfrac{u_\infty^2}{2c_p^*} - T_w\right)}{k \left.\dfrac{dT}{dy}\right|_w} = \left[1 + \frac{(\rho v)_w u_s}{\tau_w}\right]^{(c_p^c/c_p^*)(Pr^* - Pr_t)} \times \left[1 + \frac{(\rho v)_w u_\infty}{\tau_w}\right]^{(c_p^c/c_p^*)\, Pr_t}$$

or, using familiar dimensionless parameters, equation (20) of the text, with the temperature recovery factor r^* given by (21) of the text.

Résumé—On présente dans cet article une méthode d'évaluation des taux de transport d'énergie, de quantité de mouvement et de masse dans les écoulements turbulents. Elle comprend: (a) une corrélation semi-empirique du coefficient de frottement relative à l'injection de l'air dans un courant d'air à basse vitesse (elle fournit la courbe de "propriété constante" à la base de tout essai de détermination des taux de transport utilisant des états de référence); (b) des analogies de Reynolds modifiées reliant les coefficients de transport d'énergie et de masse avec le coefficient de frottement; (c) des expressions de la température de référence (ou enthalpie) et la composition de référence. Le paramètre du taux de souflage utilisé dans la corrélation du coefficient de frottement diffère des suggestions des autres auteurs basées sur les théories de sous-couches, principalement en ce qu'ils utilisent la moyenne géométrique du facteur de frottement avec et sans soufflage. Les analgoies modifiées de Reynolds diffèrent des analogies de Rubesin et Pappas principalement du fait que la chaleur spécifique du mélange est constante, sa valeur étant fixée par l'état de référence. Comme pour des écoulements sans apport de masse à la paroi, l'expression de la température de référence utilisée pour des écoulements laminaires représentent bien les résultats pour les écoulements turbulents. Pour traduire les données concernant les écoulements turbulents avec apport de masse il est conseillé d'essayer d'utiliser les expressions d'état de référence proposées par Knuth pour les écoulements laminaires avec injection de masse.

Puisqu'il n'ya actuellement aucune données concernant des mesures dépendant de la concentration du gaz injecté en surface, il n'est pas encore possible de vérifier l'analogie de Reynolds reliant le transport de masse et de quantité de mouvement et d'utiliser l'expression de concentration référence.

Les données limites valables indiquent que: (a) pour des nombres de Mach supérieurs à 3, l'expression de température référence développée 'our les écoulements laminaires avec transport de masse semble vérifier d'une façon satisfaisante les données relatives aux écoulements turbulents avec transport de masse, et (b) l'utilisation de l'analogie de Reynolds modifiée et le paramètre de soufflage proposés semble relier convenablement le coefficient de frottement et les données de transmission de chaleur dans le cas où la capacité thermique du refroidisseur et du fluide principal sont égales. On met en évidence la nécessité de réunir des données plus nombreuses et plus précises.

Zusammenfassung—Nach einer hier angegebenen Methode lassen sich Stoff-, Impuls- und Energieaustauschrate in turbulenten Strömungen ermitteln unter Berücksichtigung von (a) einer halbempirischen Beziehung für den Reibungskoeffizienten für Lufteinblasung in einen Luftstrom geringer Geschwindigkeit (um "konstante Eigenschaften" zu gewährleisten, die als Grundlage jedes Versuchs gelten, Übergangsraten mit Hilfe von Bezugszuständen zu ermitteln); (b) modifizierten Reynoldsanalogien, die Stoff- und Energieaustauschkoeffizienten mit dem Reibungskoeffizienten verbinden, und (c) Ausdrücken für die Bezugstemperatur (oder Enthalpie) und Bezugszusammensetzung. Der Parameter der für die Beziehung mit dem Reibungskoeffizienten verwendeten Einblasrate unterscheidet sich von Vorschlägen früherer Forscher, die vorwiegend Unterschichttheorien zugrundelegten darin, dass der geometrische Mittelwert des Reibungsfaktors mit Einblasung und jenes ohne Einblasung verwendet wurde. Die modifizierten Reynoldsanalogien unterscheiden sich von den Analogien von Rubesin und Pappas hauptsächlich in der Konstanz der spezifischen Wärme der Mischung mit einem vom Bezugszustand festgelegten Wert. Da für Strömungen ohne Stoffzugabe an der Wand der für Laminarströmung verwendete Ausdruck der Bezugstemperatur die Ergebnisse für turbulente Strömung gut korreliert, empfiehlt es sich, Werte für turbulenten Strom mit Stoffzusatz zu korrelieren durch Verwendung der von Knuth entwickelten Ausdrücke für den Bezugszustand für laminare Strömung mit Stoffzusatz.

Da keine zuverlässigen Messungen für Fremdgaskonzentrationen an der Oberfläche verfügbar sind ist gegenwärtig die Verifikation der modifizierten Reynoldsanalogie, die Stoff- und Impulsaustausch verbindet und der Verwendbarkeit eines Ausdruckes für die Bezugskonzentration nicht möglich. Die beschränkt verfügbaren Daten lassen erkennen, (a) dass für Machzahlen bis 3 der für laminare Strömung mit Stoffaustausch gewonnene Ausdruck für die Bezugstemperaturen die Daten für turbulente Strömung mit Stoffzugabe zufriedenstellend zu korrelieren scheint und (b) bei Verwendung der modifizierten Reynoldsanalogie und des vorgeschlagenen Parameters für die Einblasrate die Oberflächenreibung und die Wärmeübergangsdaten im Fall gleicher Wärmekapazität des Kühlmittels und des strömenden Gases zufriedenstellend zu korrelieren sind. Der Bedarf nach umfassenderen und genaueren Daten wird betont.

Аннотация—Излагается метод определения скоростей переноса массы, импульса и энергии при турбулентном течении. Метод включает в себя:

(a) полуэмпирическое соотношение для коэффициента трения при вдуве воздуха в воздушный поток малой скорости (при условии, что при построении «кривой постоянных свойств», необходимой для определения скорости переноса, используются характерные состояния),

(б) модифицированные аналогии Рейнольдса, устанавливающие зависимость между коэффициентами переноса массы и энергии и коэффициентом трения, и

(в) выражения характерных температуры (или энтальпии) и состава. Параметр скорости вдува, с помощью которого описывается коэффициент трения, отличается от параметра, предложенного ранее исследователями. Отличие заключалось в том, что вместо использования теории подслоя в данном случае берется среднегеометрическое коэффициента трения при наличии вдува и при его отсутствии. Модифицированные аналогии Рейнольдса отличаются от аналогий Рубезина и Паппаса тем, что величина удельной теплоты смеси остается той же, что и в характерном состоянии. Выражение характерной температуы для течений без подвода массы на стенке, используемое в случае ламинарных течений, как известно, весьма успешно используется для описания результатов для турбулентных течений. Поэтому рекомендуется данные для турбулентных течений с массообменом описывать с помощью выраженой характерного состояния, выведенных Кнутом для ламинарных течений с подводом массы.

В настоящее время нет надежных данных измерений концентраций инородной примеси на поверхности. Это обстоятельство делает невозможной проверку соотношений модифицированной аналогии Рейнольдса между переносом массы и импульса, а также не позволяет использовать выражение характерной концентрации.

Из имеющихся скудных данных можно сделать вывод, что:

(а) для значений числа Маха не больше 3 выражение характерной температуры, выведенное для ламинарных течений с переносом массы, удовлетворительно описывает данные для соответствующих турбулентных течений, и

(б) с помощью модифицированной аналогии Рейнольдса и предложенного параметра скорости вдуьа можно удовлетворительно описать данные по поверхностному трению и теплообмену для случая, когда теплоемкость охладителя и газа основного потока одинаковы.

Подчеркивается необходимость в более полных и точных данных.